Chapter 2. Limits and Continuity

Lecture 1. Rigorous discussion on limits

Part I. Definitions

Definition 1. Suppose $A \subset \mathbb{R}$ and $a \in \mathbb{R}$. If $\forall \delta > 0, \exists x \in A$ such that $0 < |x - a| < \delta$, then a is called a **limit point** of A

Definition 2. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $a \in \mathbb{R}$, $A \subset D$ and a is a limit point of A. If $L \in \mathbb{R}$ and $\forall \varepsilon > 0 \ \exists \ \delta > 0$ such that

$$\forall x \in A \bigcap (a - \delta, a + \delta), \ |f(x) - L| < \varepsilon$$

then we say that $f(x)$ tends (approaches) to L when x tends (approaches) to a *along* A and write

$$\lim_{A \ni x \to a} f(x) = L.$$

Remark 1. Please note that in our book "$\lim\limits_{x\to a}$" means "$\lim\limits_{a\neq x\to a}$"; "$\lim\limits_{x\to a^+}$" means "$\lim\limits_{a<x\to a}$" and "$\lim\limits_{x\to a^-}$" means "$\lim\limits_{a>x\to a}$".

Remark 2. Other sentences to express $\lim\limits_{A\ni x\to a} f(x) = L$.

a) $f(x)$ converges to L as x tends to a along A;

b) $f(x)$ has limit L as x tends to a along A.

Example 1. Let D be Dirichlet function, i.e.

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then we have for all $a \in \mathbb{R}$

$$\lim_{\mathbb{Q}\ni x\to a} D(x) = 1, \qquad \lim_{(\mathbb{R}\setminus\mathbb{Q})\ni x\to a} D(x) = 0.$$

Example 2.(in Page 140 of our book) Prove that

$$\lim_{x\to 2}(3x - 5) = 1.$$

Example 3.(in Page 141 of our book) Prove that

$$\lim_{x\to 0^+} \sqrt{x} = 0.$$

Example 4.(in Page 141 of our book) Prove that

$$\lim_{x \to 3} x^2 = 9.$$

Definition 3. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $a \in \mathbb{R}$, $A \subset D$ and a is a limit point of A. If $\forall \alpha \in \mathbb{R}\ \exists\ \delta > 0$ such that

$$\forall x \in A \bigcap (a - \delta, a + \delta),\ \ f(x) > \alpha$$

then we say that $f(x)$ tends (approaches) to $+\infty$ when x tends (approaches) to a *along* A and write

$$\lim_{A \ni x \to a} f(x) = +\infty.$$

Example 5.(in Page 144 of our book) Prove that

$$\lim_{0 \neq x \to 0} \frac{1}{x^2} = +\infty.$$

Example 6.(in Page 147 of our book, 67(b)) Prove that

$$\lim_{x \to 1^-} \frac{1}{1 - x} = +\infty.$$

Remark 3. Other sentences to express $\lim_{A \ni x \to a} f(x) = +\infty$.

a) $f(x)$ diverges to $+\infty$ as x tends to a along A;

b) $f(x)$ has limit $+\infty$ as x tends to a along A.

Definition 4. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $a \in \mathbb{R}$, $A \subset D$ and a is a limit point of A. If $\forall \alpha \in \mathbb{R}\ \exists\ \delta > 0$ such that

$$\forall x \in A \bigcap (a - \delta, a + \delta),\ \ f(x) < \alpha$$

then we say that $f(x)$ tends (approaches) to $-\infty$ when x tends (approaches) to a *along* A and write

$$\lim_{A \ni x \to a} f(x) = -\infty.$$

Example 7.(in Page 147 of our book, 67(a)) Prove that

$$\lim_{x\to 1^+} \frac{1}{1-x} = -\infty.$$

Remark 4. Other sentences to express $\lim_{A\ni x\to a} f(x) = -\infty$.

a) $f(x)$ diverges to $-\infty$ as x tends to a along A;

b) $f(x)$ has limit $-\infty$ as x tends to a along A.

Definition 5. Let $E \subset \mathbb{R}$ $(E \neq \emptyset)$. If there is a number $a \in \mathbb{R}$ such that $\forall x \in E,\ x \le a$ holds, then a is called an upper bound of E and E is called bounded above. If there is a number $b \in \mathbb{R}$ such that $\forall x \in E,\ x \ge b$ holds, then b is called a lower bound of E and E is called bounded below. If E has both upper and lower bounds then E is called bounded.

Example 8. The set $\mathbb{N}_+$ has no upper bound, i.e. for any $a \in \mathbb{R}$ there exists $n \in \mathbb{N}_+$ such that $n > a$.

Definition 6. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $A \subset D$ and A has no upper bound. If $L \in \mathbb{R}$ and $\forall \varepsilon > 0\ \exists\ N \in \mathbb{N}+$ such that

$$\forall x \in A \bigcap (N, +\infty),\ |f(x) - L| < \varepsilon$$

then we say that $f(x)$ tends (approaches) to L when x tends (approaches) to $+\infty$ *along* A and write

$$\lim_{A\ni x\to +\infty} f(x) = L.$$

(If there is a number M such that $[M, \infty) \subset A$ then $\lim_{A\ni x\to +\infty}$ may be written as $\lim_{x\to +\infty}$ simply.)

Remark 5. Other sentences to express $\lim_{A\ni x\to +\infty} f(x) = L$.

a) $f(x)$ converges to L as x tends to $+\infty$ along A;

b) $f(x)$ has limit L as x tends to $+\infty$ along A.

Example 9 Let $f(n) = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2^1} + \cdots + \frac{1}{2^n}, \quad n \in \mathbb{N}_+.$

Then

$$\lim_{\mathbb{N}_+\ni n\to +\infty} f(n) = 1.$$

Definition 7. A function defined on $\mathbb{N}_+$ is also called a sequence. A sequence f is usually denoted as $\{f(n)\}_{n=1}^{+\infty}$ for which the notation "$\lim\limits_{\mathbb{N}_+ \ni n \to +\infty} f(n)$" is usually written as "$\lim\limits_{n \to +\infty} f(n)$".

(See the book, p.649 (1))

Example 10.(in Page 143 of our book) Prove that

$$\lim_{x \to +\infty} \frac{1}{x} = 0.$$

Definition 8. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $A \subset D$ and A has no lower bound. If $L \in \mathbb{R}$ and $\forall \varepsilon > 0 \ \exists \ N \in \mathbb{N}+$ such that

$$\forall x \in A \bigcap (-\infty, -N), \ |f(x) - L| < \varepsilon$$

then we say that $f(x)$ tends (approaches) to L when x tends (approaches) to $-\infty$ *along* A and write

$$\lim_{A \ni x \to -\infty} f(x) = L.$$

(If there is a number M such that $(-\infty, M) \subset A$ then "$\lim\limits_{A \ni x \to -\infty}$" may be written as "$\lim\limits_{x \to -\infty}$"simply.)

Remark 7. Other sentences to express $\lim\limits_{A \ni x \to -\infty} f(x) = L$.

a) $f(x)$ converges to L as x tends to $-\infty$ along A;

b) $f(x)$ has limit L as x tends to $-\infty$ along A.

Definition 9. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $A \subset D$ and A has no upper bound. If $\forall \alpha \in \mathbb{R} \ \exists \ N \in \mathbb{N}+$ such that

$$\forall x \in A \bigcap (N, +\infty), \ f(x) > \alpha$$

then we say that $f(x)$ tends (approaches) to $+\infty$ when x tends (approaches) to $+\infty$ *along* A and write

$$\lim_{A \ni x \to +\infty} f(x) = +\infty.$$

Remark 8. Other sentences to express $\lim\limits_{A \ni x \to +\infty} f(x) = +\infty$.

a) $f(x)$ diverges to $+\infty$ as x tends to $+\infty$ along A;

b) $f(x)$ has limit $+\infty$ as x tends to $+\infty$ along A.

Definition 10. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $A \subset D$ and A has no lower bound. If $\forall \beta \in \mathbb{R}\ \exists\ N \in \mathbb{N}+$ such that

$$\forall x \in A \bigcap (N, +\infty)\ f(x) < \beta$$

then we say that $f(x)$ tends (approaches) to $-\infty$ when x tends (approaches) to $+\infty$ *along* A and write

$$\lim_{A \ni x \to +\infty} f(x) = -\infty.$$

Remark 9. Other sentences to express $\lim_{A \ni x \to +\infty} f(x) = -\infty$.

a) $f(x)$ diverges to $-\infty$ as x tends to $+\infty$ along A;

b) $f(x)$ has limit $-\infty$ as x tends to $+\infty$ along A.

Definition 11. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $A \subset D$ and A has no lower bound. If $\forall \alpha \in \mathbb{R}\ \exists\ N \in \mathbb{N}+$ such that

$$\forall x \in A \bigcap (-\infty, -N),\ \ f(x) > \alpha$$

then we say that $f(x)$ tends (approaches) to $+\infty$ when x tends (approaches) to $-\infty$ *along* A and write

$$\lim_{A \ni x \to -\infty} f(x) = +\infty.$$

Remark 10. Other sentences to express $\lim_{A \ni x \to -\infty} f(x) = +\infty$.

a) $f(x)$ diverges to $+\infty$ as x tends to $-\infty$ along A;

b) $f(x)$ has limit $+\infty$ as x tends to $-\infty$ along A.

Definition 12. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a non-empty set $D \subset \mathbb{R}$. Assume $A \subset D$ and A has no lower bound. If $\forall \beta \in \mathbb{R}\ \exists\ N \in \mathbb{N}+$ such that

$$\forall x \in A \bigcap (-\infty, -N)\ f(x) < \beta$$

then we say that $f(x)$ tends (approaches) to $-\infty$ when x tends (approaches) to $-\infty$ *along* A and write

$$\lim_{A \ni x \to -\infty} f(x) = -\infty.$$

Remark 11. Other sentences to express $\lim_{A\ni x\to-\infty} f(x) = -\infty$.

a) $f(x)$ diverges to $-\infty$ as x tends to $-\infty$ along A;

b) $f(x)$ has limit $-\infty$ as x tends to $-\infty$ along A.

Exercises

1. $\lim_{x\to a} \sin x = \sin a, \; a \in \mathbb{R}$.
2. $\lim_{x\to a} \cos x = \cos a, \; a \in \mathbb{R}$.
3. $\lim_{0\neq x\to 0} \frac{\sin x}{x} = 1$.
4. Assume in a same limit process $f(x)$ tends to $a \in \mathbb{R}$ and $g(x)$ tends also to a. If $\forall x, f(x) \le h(x) \le g(x)$ then $h(x)$ also tends to a. (When a is $+\infty$ or $-\infty$ the conclusion is still valid).

Part II. Arithmetic operations on limits and the limit of a composition function

Assume in a same limit process $f(x)$ tends to $a \in \mathbb{R}$ and $g(x)$ tends to $b \in \mathbb{R}$. Then

1. $\lim f(x) + g(x) = a + b$;
2. $\forall \alpha \in \mathbb{R}, \lim \alpha f(x) = \alpha a$;
3. $\lim f(x)g(x) = ab$;
4. If $a \neq 0$ then $\lim \frac{1}{f(x)} = \frac{1}{a}$.

Assume $g : A \longrightarrow U \subset B$, $f : B \longrightarrow \mathbb{R}$. Then $f \circ g$ is well defined on A. If $\lim_{A\ni x\to a} g(x) = b$, b is a limit point of B and $\lim_{B\ni y\to b} f(y) = c$ then

$$\lim_{A\ni x\to a} f \circ g(x) = c.$$

Proof. Given $\varepsilon > 0$, by the condition $\lim_{B\ni y\to b} f(y) = c$ we know that $\exists \eta > 0$ such that

$$\text{for } y \in B \bigcap (b-\eta, b+\eta) \quad |f(y) - c| < \varepsilon. \tag{1}$$

By the condition $\lim\limits_{A \ni x \to a} g(x) = b$, we know that $\exists \delta > 0$ such that

$$\text{for } x \in A \bigcap (a-\delta, a+\delta) \quad |g(x) - b| < \eta.$$

Hence

$$\text{for } x \in A \bigcap (a-\delta, a+\delta) \quad g(x) \in B \bigcap (b-\eta, b+\eta).$$

Then substituting $y = g(x)$ into (1) we get

$$\text{for } x \in A \bigcap (a-\delta, a+\delta) \quad |f(g(x)) - c| < \varepsilon. \tag{2}$$

This means

$$\lim_{A \ni x \to a} f(g(x)) = c. \qquad \square$$

Corollaries

1. If P is a polynomial, then

$$\lim_{x \to a} P(x) = P(a) \ \text{ for all } a \in \mathbb{R}.$$

2. If Q is a rational function the, then

$$\lim_{D(Q) \ni x \to a} Q(x) = Q(a) \text{ for all } a \in D(Q).$$