Hopf bifurcation in structured population dynamics

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July 2 2012
Outline

1 Motivation
- Functional differential equations
- Age-Structured Model

2 Hopf Bifurcation Theorem

3 Hopf bifurcations in age Structured Models
Several types of differential equations, such as delay differential equations, age-structured models in population dynamics, some partial differential equations, and evolution equations with nonlinear boundary conditions, can be written as semilinear Cauchy problems with non-dense domain.

\[
\begin{aligned}
\frac{dx(t)}{dt} &= Bx(t) + \hat{L}(x_t) + f(t, x_t), \forall t \geq 0, \\
\quad x_0 = \varphi \in C\left([-r, 0], \mathbb{R}^n\right),
\end{aligned}
\]

(1)

where \(x_t \in C\) satisfies \(x_t(\theta) = x(t + \theta)\), \(B \in M_n(\mathbb{R})\) is a \(n \times n\) real matrix, \(\hat{L}\) is a bounded linear operator from \(C\) to \(\mathbb{R}^n\) and has the form \(\hat{L}(\varphi) = \int_{-r}^{0} d\eta(\theta) \varphi(\theta)\), here \(\eta\) is a map of bounded variation from \([-r, 0]\) into \(M_n(\mathbb{R})\), and \(f: \mathbb{R} \times C \to \mathbb{R}^n\) is a continuous map.
FDE as Non-Densely Defined Problems:

Regard FDE as a PDE
Define \( u \in C \left( [0, +\infty) \times [-r, 0], \mathbb{R}^n \right) \) by

\[ u(t, \theta) = x(t + \theta), \forall t \geq 0, \forall \theta \in [-r, 0]. \]

Note that if \( x \in C^1 \left( [-r, +\infty), \mathbb{R}^n \right) \), then

\[ \frac{\partial u(t, \theta)}{\partial t} = x'(t + \theta) = \frac{\partial u(t, \theta)}{\partial \theta}. \]

Hence, we must have

\[ \frac{\partial u(t, \theta)}{\partial t} - \frac{\partial u(t, \theta)}{\partial \theta} = 0, \forall t \geq 0, \forall \theta \in [-r, 0]. \]
Moreover, for $\theta = 0$, we obtain
\[
\frac{\partial u(t,0)}{\partial \theta} = x'(t) = Bx(t) + \hat{L}(x_t) + f(t, x_t)
\]
\[
= Bu(t,0) + \hat{L}(u(t,\cdot)) + f(t, u(t,\cdot)), \forall t \geq 0.
\]

Therefore, we deduce formally that $u$ must satisfy a PDE
\[
\begin{align*}
\frac{\partial u(t, \theta)}{\partial t} - \frac{\partial u(t, \theta)}{\partial \theta} &= 0, \\
\frac{\partial u(t, 0)}{\partial \theta} &= Bu(t,0) + \hat{L}(u(t,\cdot)) + f(t, u(t,\cdot)), \forall t \geq 0, \\
u(0,\cdot) &= \varphi \in C \left([-r, 0], \mathbb{R}^n\right).
\end{align*}
\]

(2) Zhihua Liu SMC BNU Hopf bifurcation in structured population dynamics
FDE as Non-Densely Defined Problems:

Regard PDE as an abstract non-densely defined Cauchy problem.

\[ X = \mathbb{R}^n \times C([-r, 0], \mathbb{R}^n) \]

taken with the usual product norm

\[ \left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\| = \| x \|_{\mathbb{R}^n} + \| \varphi \|_{\infty}. \]

Define the linear operator \( A : D(A) \subset X \to X \) by

\[ A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}, \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A), \]

with

\[ D(A) = \{ 0_{\mathbb{R}^n} \} \times C^1([-r, 0], \mathbb{R}^n). \]

Note that \( A \) is non-densely defined because

\[ \overline{D(A)} = \{ 0_{\mathbb{R}^n} \} \times C([-r, 0], \mathbb{R}^n) \neq X. \]
FDE as Non-Densely Defined Problems:

We also define \( L : \overline{D(A)} \to X \) by

\[
L \left( \begin{array}{c}
0_{\mathbb{R}^n} \\
\varphi
\end{array} \right) = \left( \begin{array}{c}
\hat{L}(\varphi) \\
0_C
\end{array} \right).
\]

and \( F : \mathbb{R} \times \overline{D(A)} \to X \) by

\[
F \left( t, \left( \begin{array}{c}
0_{\mathbb{R}^n} \\
\varphi
\end{array} \right) \right) = \left( \begin{array}{c}
f(t, \varphi) \\
0_C
\end{array} \right).
\]

By setting \( v(t) = \left( \begin{array}{c}
0_{\mathbb{R}^n} \\
u(t)
\end{array} \right) \), we obtain

\[
\frac{dv(t)}{dt} = Av(t) + L(v(t)) + F(t, v(t)), \quad t \geq 0, \quad v(0) = \left( \begin{array}{c}
0_{\mathbb{R}^n} \\
\varphi
\end{array} \right) \in \overline{D(A)}.
\]
Age-Structured Model

\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} &= -D(a)u(t, a) + M(\mu, u(t, .))(a), \quad a \geq 0, \ t \geq 0, \\
u(t, 0) &= B(\mu, u(t, .)) \\
u(0, .) &= u_0 \in L^1((0, +\infty), \mathbb{R}^n),
\end{align*}

where \( \mu \in \mathbb{R} \),
\( D(.) = \text{diag}(d_1(.), ..., d_n(.)) \in L^\infty ((0, +\infty), M_n(\mathbb{R}^+)) \),
\( M : \mathbb{R} \times L^1((0, +\infty), \mathbb{R}^n) \to L^1((0, +\infty), \mathbb{R}^n) \), and
\( B : \mathbb{R} \times L^1((0, +\infty), \mathbb{R}^n) \to \mathbb{R}^n \) are \( k \)-time continuously differentiable.
Consider the Banach space

\[ X = \mathbb{R}^n \times L^1 ((0, +\infty), \mathbb{R}^n), \]

the linear operator \( A : D(A) \subset X \to X \) defined by

\[
A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - D\varphi \end{pmatrix}
\]

with

\[ D(A) = \{0\} \times W^{1,1} ((0, +\infty), \mathbb{R}^n). \]

We observe that \( A \) is non-densely defined since

\[ \overline{D(A)} = \{0\} \times L^1 ((0, +\infty), \mathbb{R}^n) \neq X. \]
Define the function $F : \mathbb{R} \times \overline{D(A)} \rightarrow X$ by

$$F \left( \mu, \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \right) = \begin{pmatrix} B(\mu, \varphi) \\ M(\mu, \varphi) \end{pmatrix}.$$ 

Setting $v(t) = \begin{pmatrix} 0 \\ u(t, \cdot) \end{pmatrix}$, we can rewrite the age structured model as the following non-densely defined abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + F(\mu, v(t)), \quad t \geq 0, \quad v(0) = \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \in \overline{D(A)}.$$
Outline

1. Motivation
   - Functional differential equations
   - Age-Structured Model

2. Hopf Bifurcation Theorem

3. Hopf bifurcations in age Structured Models
Hopf Bifurcation Theorem

Zhihua Liu, Pierre Magal and Shigui Ruan, Hopf bifurcation for non-densely defined Cauchy problems, ZAMP 62(2011) 191-222.

\[
\frac{du(t)}{dt} = Au(t) + F(\mu, u(t)), \quad \forall t \geq 0, \quad u(0) = x \in D(A),
\]

(4)

where \( A : D(A) \subset X \to X \) is a linear operator on a Banach space \( X \), \( F : \mathbb{R} \times D(A) \to X \) is \( C^k \) map with \( k \geq 2 \), and \( \mu \in \mathbb{R} \) is the bifurcation parameter. Here, \( D(A) \) is not dense in \( X \) and \( A \) is not necessarily a Hille-Yosida operator. Also the solutions must be understood as integrated solutions of (4), that is,

\[
\int_0^t u(s)ds \in D(A), \quad \forall t \geq 0,
\]

and

\[
u(t) = x + A \int_0^t u(s)ds + \int_0^t F(\mu, u(s))ds, \quad \forall t \geq 0.\]
Recently, some situation where $A$ is not a Hille-Yosida operator has been studied (see works by Magal and Ruan (P. Magal and S. Ruan, On integrated semigroups and age structured models in $L^p$ spaces, Differential Integral Equations 20 (2007), 197-139) and Thieme (H.R. Thieme, Differentiability of convolutions, integrated semigroups of bounded semi-variation, and the inhomogeneous Cauchy problem, J. Evol. Equ. 8 (2008), 1–23).
Hopf Bifurcation Theorem

Recall.

(◊) $A$ is a Hille-Yosida operator if there exist two constants $\omega_A \in \mathbb{R}$ and $M_A \geq 1$ such that $(\omega_A, +\infty) \subset \rho(A)$ and

$$\| (\lambda I - A)^{-k} \|_{\mathcal{L}(X)} \leq \frac{M_A}{(\lambda - \omega_A)^k}, \forall \lambda > \omega_A, \forall k \geq 1,$$

where $\mathcal{L}(X)$ is the space of bounded linear operators from $X$ into $X$ and $\rho(A)$ is the resolvent set of $A$.

(◊) $A_0 : D(A_0) \subset X_0 := \overline{D(A)} \to X_0$, the part of $A$ in $X_0$, is the linear operator on $X_0$ defined by

$$A_0 x = A x, \forall x \in D(A_0) = \left\{ x \in D(A) : A x \in \overline{D(A)} \right\}.$$
Here we assume that A satisfies some weaker conditions.

**Assumption 1.** Assume that $A : D(A) \subset X \to X$ is a linear operator on a Banach space $(X, \| \cdot \|)$ such that there exist two constants, $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that $(\omega_A, +\infty) \subset \rho(A)$ and the following properties are satisfied:

(a) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \forall x \in X$;

(b) $\| (\lambda I - A)^{-k} \|_{\mathcal{L}(X_0)} \leq \frac{M_A}{(\lambda - \omega_A)^k}, \forall \lambda > \omega_A, \forall k \geq 1$. 

Zhihua Liu SMC BNU Hopf bifurcation in structured population dynamics
**Assumption 2.** There exists a function \( \delta : [0, +\infty) \to [0, +\infty) \) with

\[
\lim_{t(>0) \to 0} \delta(t) = 0,
\]

such that for each \( \tau > 0 \) and \( f \in C ([0, \tau], X) \),

\[
t \to \int_0^t S_A(t - s)f(s)ds
\]

is continuously differentiable and

\[
\left\| \frac{d}{dt} \int_0^t S_A(t - s)f(s)ds \right\| \leq \delta(t) \sup_{s \in [0,t]} \| f(s) \|, \forall t \in [0, \tau].
\]
Recently, a center manifold reduction theory has been developed by Magal and Ruan (P. Magal and S. Ruan, Center manifolds for semilinear equations with non-dense domain and applications on Hopf bifurcation in age structured models, Mem. Amer. Math. Soc. 202 (2009), No. 951) for non-densely defined Cauchy problems satisfying Assumption 1 and Assumption 2. The following Hopf bifurcation theorem is proved by using the center manifold theory for non-densely defined Cauchy problems associated with the integrated semigroup theory.
Motivation

Hopf Bifurcation Theorem

Hopf bifurcations in age Structured Models

Hopf Bifurcation Theorem

Recall

Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear $C^0$-semigroup $\{T_L(t)\}_{t \geq 0}$ on a Banach space $X$. We define $\omega_0(L) \in [-\infty, +\infty)$ the growth bound of $L$ by

$$\omega_0(L) := \lim_{t \to +\infty} \frac{\ln \left( \|T_L(t)\|_{L(X)} \right)}{t}.$$

The essential growth bound $\omega_{0,\text{ess}}(L) \in [-\infty, +\infty)$ of $L$ is defined by

$$\omega_{0,\text{ess}}(L) := \lim_{t \to +\infty} \frac{\ln \left( \|T_L(t)\|_{\text{ess}} \right)}{t},$$

where $\|T_L(t)\|_{\text{ess}}$ is the essential norm of $T_L(t)$ defined by

$$\|T_L(t)\|_{\text{ess}} = \kappa \left( T_L(t)B_X(0, 1) \right),$$

here $B_X(0, 1) = \{x \in X : \|x\|_X \leq 1\}$. 
Assumption 3. Let $\varepsilon > 0$ and $F \in C^k ((-\varepsilon, \varepsilon) \times B_{X_0} (0, \varepsilon); X)$ for some $k \geq 4$. Assume that the following conditions are satisfied:

(a) $F (\mu, 0) = 0, \forall \mu \in (-\varepsilon, \varepsilon)$, and $\partial_x F (0, 0) = 0$.

(b) The essential growth rate of $\{T_{A_0} (t)\}_{t \geq 0}$ is strictly negative, that is,

$$\omega_{0, \text{ess}} (A_0) < 0.$$
(c) (Transversality condition) For each \( \mu \in (-\varepsilon, \varepsilon) \), there exists a pair of conjugated simple eigenvalues of 
\((A + \partial_x F(\mu, 0))_0\), denoted by \( \lambda(\mu) \) and \( \overline{\lambda(\mu)} \), such that

\[
\lambda(\mu) = \alpha(\mu) + i\omega(\mu),
\]

the map \( \mu \rightarrow \lambda(\mu) \) is continuously differentiable,

\[
\omega(0) > 0, \quad \alpha(0) = 0, \quad \frac{d\alpha(0)}{d\mu} \neq 0,
\]

and

\[
\sigma(A_0) \cap i\mathbb{R} = \left\{ \lambda(0), \overline{\lambda(0)} \right\}.
\]
Theorem (Hopf Bifurcation)

Let Assumptions 1-3 be satisfied. Then there exist $\varepsilon^* > 0$, three $C^{k-1}$ maps, $\varepsilon \to \mu(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, $\varepsilon \to x_\varepsilon$ from $(0, \varepsilon^*)$ into $D(A)$, and $\varepsilon \to \gamma(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, such that for each $\varepsilon \in (0, \varepsilon^*)$ there exists a $\gamma(\varepsilon)$-periodic function $u_\varepsilon \in C^k(\mathbb{R}, X_0)$, which is an integrated solution of (4) with the parameter value equals $\mu(\varepsilon)$ and the initial value equals $x_\varepsilon$. So for each $t \geq 0$, $u_\varepsilon$ satisfies

$$u_\varepsilon(t) = x_\varepsilon + A \int_0^t u_\varepsilon(l) dl + \int_0^t F(\mu(\varepsilon), u_\varepsilon(l)) dl.$$
Moreover, we have the following properties

(1) There exist a neighborhood $N$ of 0 in $X_0$ and an open interval $I$ in $\mathbb{R}$ containing 0, such that for $\hat{\mu} \in I$ and any periodic solution $\hat{u}(t)$ in $N$ with minimal period $\hat{\gamma}$ close to $\frac{2\pi}{\omega(0)}$ of (4) for the parameter value $\hat{\mu}$, there exists $\varepsilon \in (0, \varepsilon^*)$ such that $\hat{u}(t) = u_{\varepsilon}(t + \theta)$ (for some $\theta \in [0, \gamma(\varepsilon)]$), $\mu(\varepsilon) = \hat{\mu}$, and $\gamma(\varepsilon) = \hat{\gamma}$. 
Hopf Bifurcation Theorem

(2) The map \( \varepsilon \to \mu(\varepsilon) \) is a \( C^{k-1} \) function and we have the Taylor expansion

\[
\mu(\varepsilon) = \sum_{n=1}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \mu_{2n} \varepsilon^{2n} + O(\varepsilon^{k-1}), \quad \forall \varepsilon \in (0, \varepsilon^*),
\]

where \( \left\lfloor \frac{k-2}{2} \right\rfloor \) is the integer part of \( \frac{k-2}{2} \).

(3) The period \( \gamma(\varepsilon) \) of \( t \to u_\varepsilon(t) \) is a \( C^{k-1} \) function and

\[
\gamma(\varepsilon) = \frac{2\pi}{\omega(0)} [1 + \sum_{n=1}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \gamma_{2n} \varepsilon^{2n}] + O(\varepsilon^{k-1}), \quad \forall \varepsilon \in (0, \varepsilon^*),
\]

where \( \omega(0) \) is the imaginary part of \( \lambda(0) \) defined in Assumption 3.
The main idea of the proof
(a) Incorporate the parameter into the state variable

\[
\begin{align*}
\frac{d\mu(t)}{dt} &= 0 \\
\frac{du(t)}{dt} &= Au(t) + F(\mu(t), u(t)) \\
(\mu(0), u(0)) &= (\mu_0, u_0) \in (-\epsilon, \epsilon) \times D(A).
\end{align*}
\]

(b) Rewrite the above system as the following abstract Cauchy problem

\[
\frac{dv(t)}{dt} = Av(t) + F(v(t)), \text{ for } t \geq 0,
\]

with \(v(0) = v_0 \in \overline{D(A)}\).
(c) In order to apply the center manifold theory to system (5), we study the spectral properties of the linear operator $A$ and obtain a state space decomposition with respect to the spectral properties of the linear operator $A$.

(d) By applying the center manifold theory to system (5), we obtain a reduced system and prove we can apply the Hopf bifurcation theorem in the book by Hassard to the reduced system.
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2 Hopf Bifurcation Theorem

3 Hopf bifurcations in age Structured Models
Hopf bifurcation of an age-structured compartmental pest-pathogen model

Zhen Wang and Zhihua Liu, JMAA, 385(2012), 1134-1150.

\[
\begin{align*}
S'(t) &= \delta S(t) - \alpha_1 P(t)S(t), \quad t > 0, \\
\frac{\partial l(t, a)}{\partial t} + \frac{\partial l(t, a)}{\partial a} &= -d_l(a)l(t, a), \quad t, a > 0, \\
P'(t) &= \gamma - d_p P(t) + \int_0^{+\infty} \beta(a)l(t, a)da - \alpha_2 P(t)S(t), \quad t, a > 0, \\
l(t, 0) &= \alpha_1 P(t)S(t), \quad t > 0 \\
S(0) &= S_0 \geq 0, \quad P(0) = P_0 \geq 0, \quad l(0, a) = l_0(a) \in L^1_+ ((0, +\infty), \mathbb{R}),
\end{align*}
\]
Hopf bifurcation of an age-structured compartmental pest-pathogen model

where \( \delta = r - b_0 \),
\( b_0 > 0 \) is the host death rate independent of the pathogen,
\( r > 0 \) is the per capita host birth rate.
\( \alpha_1 > 0 \) denotes the number of susceptible pests converted to infected pests per pathogen cell and unit of time,
\( \alpha_2 > 0 \) represents the number of pathogen consumed to propagate the infection per susceptible pest and unit of time,
\( d_p > 0 \) is the rate of pathogen removed,
\( d_I(a) \) represents the rate of infected hosts removed with age of infection \( a \),
\( \gamma \) is the rate of introduction of the pathogen in a biological control program,
\( \beta(a) \) denotes the rate at which the infected pests release pathogen.
Assume that

\[ d_I(a) := \begin{cases} 
   d^*, & \text{if } a \geq \tau, \\
   0, & \text{if } a \in (0, \tau), 
\end{cases} \]

\[ \beta(a) := \begin{cases} 
   \beta^*, & \text{if } a \geq \tau, \\
   0, & \text{if } a \in (0, \tau), 
\end{cases} \]

where \( \tau > 0 \) and \( d^* > 0, \beta^* > 0 \).
Hopf bifurcation of an age-structured compartmental pest-pathogen model

The main result in this paper is the following theorem.

**Theorem** Let Assumptions above and \( a(b + e) > c + d > 0 \) be satisfied. Assume \( \delta > 0, \alpha_1 \beta^* - \alpha_2 d^* > 0, d_p \delta - \alpha_1 \gamma > 0 \). Then there exist \( \tau_k > 0, k = 0, 1, 2, \ldots \), such that the age-dependent compartmental pest-pathogen model undergoes a Hopf bifurcation at the equilibrium \( (I_{\tau_k}(a), S, P) \). In particular, a non-trivial periodic solution bifurcates from the equilibrium \( (I_{\tau_k}(a), S, P) \) when \( \tau = \tau_k \).

where
\[
\begin{align*}
a &= d^* + d_p + \alpha_2 S > 0, \\
b &= (d^* - \delta) \alpha_2 S + d^* d_p, \\
c &= -\delta d^* \alpha_2 S < 0, \\
d &= \delta \beta^* \alpha_1 S > 0, \\
e &= -\beta^* \alpha_1 S < 0.
\end{align*}
\]
Fig1. We choose initial values $S(0) = 9.2$, $P(0) = 0.8$, $I(0, a) = 1.472e^{(-0.5a)}$. A1-A3: $\tau' = 0.15$, B1-B3: $\tau'' = 1$. 
Hopf Bifurcation Theorem

Hopf bifurcations in age Structured Models

Motivation

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Hopf bifurcation in structured population dynamics
Hopf Bifurcation in a Size Structured Population Dynamic Model with Random Growth


\[
\begin{align*}
\frac{\partial u(t, x)}{\partial t} + \frac{\partial u(t, x)}{\partial x} &= \varepsilon^2 \frac{\partial^2 u(t, x)}{\partial x^2} - \mu u(t, x), \quad t \geq 0, x \geq 0, \\
-\varepsilon^2 \frac{\partial u(t, 0)}{\partial x} + u(t, 0) &= \alpha h(\int_0^{+\infty} \gamma(x) u(t, x) dx), \\
u(0, .) &= u_0 \in L_+^1 (0, +\infty).
\end{align*}
\]

where \( u(t, x) \) represents the population density of certain species at time \( t \) with size \( x \), \( g > 0, \varepsilon \geq 0, \mu > 0, \alpha > 0, \gamma \in L_+^{\infty} (0, +\infty) \setminus \{0\} \) and the map \( h: \mathbb{R} \to \mathbb{R} \) is defined by

\[
h(x) = xe^{-\xi x}, \forall x \geq 0.
\]
Thank you!